THE FANO OF LINES, THE KUZNETSOV COMPONENT, AND A FLOP

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ABSTRACT. The Kuznetsov component of the derived category of a cubic fourfold is a 'non-commutative K3 surface'. Its symmetric square is hence a 'non-commutative hyperkähler fourfold'. We prove that this category is equivalent to the derived category of an actual hyperkähler fourfold: the Fano of lines in the cubic. This verifies a conjecture of Galkin.

One of the key steps in our proof is a new derived equivalence for a specific 12-dimensional flop.

1. INTRODUCTION

Let $Y \subset \mathbb{P}^5$ be a smooth cubic 4-fold. There is an associated Fano variety F_Y of lines in Y, which is a smooth hyperkähler 4-fold. More categorically, there is an associated Kuznetsov component

$$\mathcal{A}_Y \subset D^b(Y)$$

which famously behaves as a 'non-commutative K3 surface'. For special classes of cubics this category $\mathcal{A}_Y$ is actually equivalent to $D^b(S)$ for some genuine K3 surface S, and in these cases there is another hyperkähler 4-fold available: the Hilbert scheme $S^{[2]}$. So it is reasonable to suppose that these two hyperkähler 4-folds might be related.

Moreover, if we are only interested in categories, then there is no need to assume that $\mathcal{A}_Y$ is geometric. If S does exist then by [BKR] we have

$$D^b(S^{[2]}) \cong D^b_{\mathbb{Z}_2}(S \times S) \cong \operatorname{Sym}^2 \mathcal{A}_Y$$

where the latter operation – taking the symmetric square of a category – is a purely formal operation. So for any Y we can imagine there would be a relationship between $\operatorname{Sym}^2 \mathcal{A}_Y$ and the derived category of F_Y. And indeed there is.

Theorem A. For any smooth cubic 4-fold Y we have an equivalence:

$$D^b(F_Y) \cong \operatorname{Sym}^2 \mathcal{A}_Y$$

This result was apparently conjectured first by Galkin,[1] and a weaker version of the statement was proven by Belmans-Fu-Raeschelders [BFR]. We learned about it from a recent paper of Bottini-Huybrechts [BH] who prove it for certain classes of cubics, see that paper for a nice outline of the history and also a discussion of the Hodge theory (which we will not touch on here). Some related conjectures are surveyed in [BFM].[2]

We prove Theorem A by using some well-established tricks in the world of matrix factorizations to reduce it to a simpler piece of higher-dimensional geometry. Let

$$X_0 \subset \operatorname{Sym}^3(\mathbb{C}^6)^\vee$$

denote the space of cubic forms with a 4-dimensional kernel, *i.e.* which can be factored through $\operatorname{Sym}^3 T^\vee$ for some 2-dimensional quotient T of $\mathbb{C}^6$. This is a singular 12-fold. It has an obvious crepant resolution, which we call X_-, given by the total space of the vector bundle $\operatorname{Sym}^3 T^\vee$ over $\operatorname{Gr}(6,2)$. As we shall see, X_0 also has a second crepant resolution X_+, which is a smooth orbifold.

[1] After completing this paper we were made aware of a 2017 talk by Galkin in which he not only states the conjecture but also explicitly anticipates the outline of our proof, see Remark 2.11.

[2] In the language of that paper, our result proves (**FM**) $\Longrightarrow$ (**DF**).

Theorem B. X_+ and X_- are derived equivalent.

We explain the argument connecting Theorems A and B in Section 2. It follows a similar pattern to various other examples, see for example [ADS, Sect. 1.3].

Theorem B (Theorem 2.5 below) is a special case of the well-known Bondal-Orlov-Kawamata conjecture that flops should induce derived equivalences. It may be of independent interest as it does not fall immediately to existing techniques, we have to do a bit of extra work. Thus it provides one more data point for the general conjecture.

Acknowledgements. We thank Daniel Huybrechts for encouraging correspondence and for pointing us to [G].

2. From B to A

Fix a non-singular cubic polynomial f in 6 variables, so $Y = (f) \subset \mathbb{P}^5$ is a smooth cubic 4-fold. By Orlov's theorem [O2, S] the Kuznetsov component $\mathcal{A}_Y \subset D^b(Y)$ is equivalent to a category of matrix factorizations on an orbifold:[3]

$$\mathcal{A}_Y \;\cong\; \mathrm{MF}\big(\,[\mathbb{C}^6/\mathbb{Z}_3],\, f\,\big)$$

It follows that the symmetric square of $\mathcal{A}_Y$ is given by matrix factorizations on the symmetric square of this orbifold, *i.e.*

$$\mathrm{Sym}^2 \mathcal{A}_Y \;\cong\; \mathrm{MF}\big(\,[(\mathbb{C}^6 \times \mathbb{C}^6)/\Gamma],\, f_1 + f_2\,\big) \tag{2.1}$$

where Γ is the semidirect product:

$$\Gamma = (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2 \tag{2.2}$$

Now consider the Fano variety of lines F_Y. This is a subvariety of $\mathrm{Gr}(2,6)$ cut out by a transverse section of $\mathrm{Sym}^3 U^\vee$, where U is the tautological rank 2 bundle. By Knörrer periodicity ([O1] *etc.*), we have an equivalence

$$D^b(F_Y) \;\cong\; \mathrm{MF}(X_-, W) \tag{2.3}$$

where X_- is the total space of the dual bundle

$$\mathrm{Sym}^3 U \to \mathrm{Gr}(2,6)$$

and W is the superpotential induced by f. We can build this space X_- as a GIT quotient of the vector space

$$\mathrm{Hom}(U, \mathbb{C}^6) \oplus \mathrm{Sym}^3 U$$

by the group $GL(U) \cong GL_2$. If we write (Φ, ψ) for the coordinates on the two summands, then the superpotential is:

$$W = (f \circ \mathrm{Sym}^3 \Phi)(\psi)$$

Note that this GIT problem is 'Calabi-Yau' because GL_2 is acting trivially on the determinant of the vector space. Consequently X_- is a (non-compact) Calabi-Yau, *i.e.* it is a crepant resolution of the underlying quotient singularity.

Since it is more comfortable to think in terms of cubic polynomials rather than symmetric trivectors, we replace U with $T = U^\vee$, so our GIT problem is the stack:

$$\mathfrak{X} = \big[\,\mathrm{Hom}(\mathbb{C}^6, T) \oplus \mathrm{Sym}^3 T^\vee \;/\; GL(T)\,\big] \tag{2.4}$$

This GIT problem has two possible stability conditions. For one choice, the semistable locus is $\{\mathrm{rank}\, \Phi = 2\}$ and the GIT quotient is X_-. We claim (see Section 3.3) that for the other stability condition, semistability is the following two conditions:

- The cubic ψ is non-zero with at least two distinct roots.
- The image of Φ is not contained in a double root of ψ.

[3]This statement requires an additional R-charge/grading on the orbifold which acts diagonally with weight 1/3. This R-charge is present throughout our constructions but it plays no real role in the arguments so we leave it to the reader to insert it where needed.

This other GIT quotient, which we denote X_+, is a smooth Calabi-Yau orbifold. Thus X_+ and X_- are two crepant resolutions of the quotient singularity X_0 underlying the stack $\mathcal{X}$, and we have a 12-dimensional flop:

$$X_- \dashleftarrow\dashrightarrow X_+$$

As explained in the introduction, the Bondal-Orlov-Kawamata conjecture predicts that this flop should induce a derived equivalence, and our Theorem B verifies this. More precisely we will prove:

Theorem 2.5. *There are tilting generators $E_- \in D^b(X_-)$ and $E_+ \in D^b(X_+)$ with the same endomorphism algebra*

$$A = \mathrm{End}_{X_-}(E_-) = \mathrm{End}_{X_+}(E_+)$$

and hence $D^b(X_-) \cong D^b(A) \cong D^b(X_+)$.

Remark 2.6.
1. The object E_- is the 'obvious' tilting vector bundle coming from Kapranov's exceptional collection on the Grassmannian. But the object E_+ is not quite a vector bundle, and this is where some extra work is necessary.
2. The algebra A is a non-commutative crepant resolution of X_0 and an example of the general theory of [ŠVdB].

The superpotential W on $\mathcal{X}$ restricts to give a superpotential on both X_- and X_+. We can also view it as a central element of the algebra A. Then the following corollary is standard (*e.g.* [S, Lem. 3.6].

Corollary 2.7.

$$\mathrm{MF}(X_-, W) \;\cong\; \mathrm{MF}(A, W) \;\cong\; \mathrm{MF}(X_+, W)$$

Now let $X_+^o \subset X_+$ be the open substack where the cubic ψ has distinct roots. The group GL_2 acts transitively on such cubics, thus in this locus we may fix ψ to be

$$\psi = t_1^3 + t_2^3$$

and we are left with a residual action of the stabilizer of this cubic polynomial, which is the finite group Γ (2.2). So

$$X_+^o = [\,\mathbb{C}^6 \times \mathbb{C}^6 \,/\, \Gamma\,] \tag{2.8}$$

and moreover the superpotential W in this locus is the function $f_1 + f_2$. This is precisely the LG model (2.1) that describes $\mathrm{Sym}^2 \mathcal{A}_Y$.

Finally we claim (see Section 3.5) that the critical locus of W on X_+ is entirely contained in the open set X_+^o, which implies that the restriction functor

$$\mathrm{MF}(X_+, W) \to \mathrm{MF}(X_+^o, W) \tag{2.9}$$

is an equivalence [O1]. So Corollary 2.7 provides the central link in a chain of equivalences

$$D^b(F_Y) \;\cong\; \mathrm{MF}(X_-, W) \;\cong\; \mathrm{MF}(X_+, W) \;\cong\; \mathrm{MF}(X_+^o, W) \;\cong\; \mathrm{Sym}^2 \mathcal{A}_Y$$

and proves Galkin's conjecture.

Remark 2.10. The smoothness of Y is required only for the step where we restrict to X_-^o, *i.e.* to have that (2.9) is an equivalence. The other steps work for all cubics f.

Remark 2.11. In [G], Galkin points out the first and last of the above equivalences and observes that X_- and X_+^o are related by the GIT problem $\mathcal{X}$. Our contribution is to understand the role played by X_+.

3. Supporting details

3.1. **Cubics.** The space $\mathrm{Sym}^3 T^\vee$ of cubics in two variables has an obvious stratification

$$\{0\} = \Sigma_0 \subset \Sigma_1 \subset \Sigma_2 \subset \Sigma_3 = \mathrm{Sym}^3 T^\vee$$

by the number of roots of the cubic. Within each $\Sigma_i \setminus \Sigma_{i-1}$ the GL_2 action is transitive and it will be convenient at times to use the following 'standard forms' for a cubic ψ:

$$t_1^3, \qquad t_1^2 t_2, \qquad t_1^3 + t_2^3$$

Note that Σ_2 is a divisor, cut out by the discriminant polynomial of a cubic. The stratum Σ_1 is a surface.

3.2. **Representation theory.** Recall that Γ is the finite subgroup (2.2) of $GL(T) \cong GL_2(\mathbb{C})$ that stabilizes the cubic $t_1^3 + t_2^3$. For later use, we collect some elementary facts on the representation theory of Γ:

(i) Γ has nine irreducible representations, of which three are 2-dimensional and six are 1-dimensional.
(ii) The restriction of $\det(T)$ to Γ generates the group of characters $\mathrm{Pic}(B\Gamma) \cong \mathbb{Z}_6$.
(iii) The restriction of $\mathrm{Sym}^2 T \otimes (\det T)^{-1}$ to Γ contains a 1-dimensional summand which is the restriction of $\det(T)^3$.

3.3. **Stability.** Here we analyse GIT stability on the stack $\mathcal{X}$ (2.4). There are two stability conditions given by either a positive or negative power of the character det of $GL(T)$.

As usual we look for destabilizing 1-parameter subgroups. The most obvious is the centre

$$\lambda_0 = \{t1_T,\, t \in \mathbb{C}^*\}$$

which destabilizes either the locus $\Phi = 0$ or the locus $\psi = 0$. For the remainder we fix a basis of T, and seek 1-parameter subgroups which fix more than just the origin. Up to conjugation and scaling there are exactly two.

- $\lambda_1 = (0, 1)$. For the negative stability condition, this destabilizes points where $\mathrm{rank}\,\Phi < 2$. For the positive stability condition, it destablizes points where $\psi \in \Sigma_1$, *i.e.* cubics with a triple root.
- $\lambda_2 = (-1, 2)$. For the negative stability condition this destabilizes the set:

  $$\{\mathrm{rank}\,\Phi < 2 \text{ and the image of } \Phi \text{ is a root of } \psi\},$$

  but these points are already destabilized by λ_1. For the positive stability condition, it destabilizes:

  $$\{\mathrm{rank}\,\Phi < 2 \text{ and the image of } \Phi \text{ is a double root of } \psi\}.$$

Thus, for the negative stability condition the unstable locus is exactly $\{\mathrm{rank}\,\Phi < 2\}$. There are no strictly semistable points, and the GIT quotient is the vector bundle

$$X_- = \mathrm{Tot}\{\ \mathrm{Sym}^3 T^\vee \to \mathrm{Gr}(6, 2)\ \}$$

as claimed above. For the positive stability condition, we see that:

- Points with $\psi \in \Sigma_1$ are unstable.
- Points with $\psi \in \Sigma_3 \setminus \Sigma_2$ are stable.
- Points with $\psi \in \Sigma_2 \setminus \Sigma_1$ are unstable if and only if the image of Φ lies in the double root.

The GIT quotient X_+ contains an open set X_+^o where $\psi \notin \Sigma_2$, this is equivalent to $[\mathbb{C}^{12}/\Gamma]$ as noted above (2.8). Now consider the divisor:

$$D = X_+ \setminus X_+^o$$

Within D we can set $\psi = t_1^2 t_2$, so the residual symmetry group is exactly λ_2. Semistability requires that the image of Φ does not lie in the double root $\{t_1 = 0\}$, *i.e.* the top row of the matrix is non-zero. It follows that

$$D \cong \mathrm{Tot}\left\{\ \mathcal{O}(-2)^{\oplus 6} \to \mathbb{P}^5\ \right\} \tag{3.1}$$

We observe that there are no semistable points with infinite isotropy, so X_+ is a smooth orbifold.

Remark 3.2. D is cut out by the discriminant of the cubic ψ, which is a degree 4 polynomial in the coefficients of ψ. Since $\psi \in \mathrm{Sym}^3 T^\vee$ its discriminant lies in $\mathrm{Sym}^4(\mathrm{Sym}^3 T)$, specifically in the 1-dimensional summand $(\det T)^6$. Hence $\mathcal{O}(D) = (\det T)^6$.

3.4. **The flop.** We now make a few more observations on the geometry of the flop between X_+ and X_-. They are not all logically necessary for our argument, but we found them reassuring.

The quotient singularity underlying $\mathcal{X}$ is, as stated in the introduction, the subvariety

$$X_0 \subset \mathrm{Sym}^3(\mathbb{C}^6)^\vee$$

of cubic forms which have "rank 2". Obviously X_0 is singular at the origin, but this singularity is not isolated. There is a non-compact locus of singularities

$$X_0^{sg} \subset X_0$$

consisting of the cubics which are "rank 1", *i.e.* cubics that are a cube of a linear form.

The spaces X_+ and X_- are crepant resolutions of the Gorenstein quotient singularity X_0. We claim that away from the origin they are related by a family of standard toric flops.

To see this, fix a generic point in X_0^{sg}, which we can take to be the cubic x_1^3. The deformations of this cubic that stay in X_0^{sg} to first order are given by adding the terms x_1^3 and $x_1^2 x_i$ for $i = 2, .., 6$. So, a transverse slice to X_0^{sg} is given by the family

$$x_1^3 + x_1 Q(x_2, ..., x_6) + C(x_2, ..., x_6)$$

for a general quadratic polynomial Q and cubic C. Now consider the preimage of this family in $\mathcal{X}$. We must have that the first basis vector is not in the kernel of Φ, so we can take the first column of Φ to be $\binom{1}{0}$, so ψ is of the form:

$$\psi = t_1^3 + \alpha t_1 t_2^2 + \beta t_2^3$$

The residual symmetry is the 1-parameter subgroup $\lambda_1 = (0, 1) \subset GL_2$, which acts on the second row of Φ with weights 1 and on α, β with weights -2 and -3. In X_-, semistability forces the second row of Φ to be non-zero, and in X_+, semistability forces $(\alpha, \beta) \neq (0, 0)$. So in this locus, our flop is a trivial family, over the base $\mathbb{A}^5$, of the standard orbifold flop:

$$\mathrm{Tot}\left\{\ \mathcal{O}(-2) \oplus \mathcal{O}(-3) \to \mathbb{P}^4\ \right\} \quad \dashleftarrow\dashrightarrow \quad \mathrm{Tot}\left\{\ \mathcal{O}(-1)^{\oplus 5} \to \mathbb{P}^1_{2:3}\ \right\}$$

Now we examine what happens at the origin in X_0. The fibre over the origin in X_- is the Grassmannian $\mathrm{Gr}(6, 2)$. On the positive side, let us denote the fibre over the origin by:

$$F \subset X_+$$

Then F is the locus in X_+ where the image of Φ lies in a root of ψ. It's not immediately obvious what this space is, but we can understand it by intersecting with the open set X_+^o and the complementary divisor D (3.1). The intersection of F with D is just the zero section:

$$F \cap D = \mathbb{P}^5$$

The intersection of F with X_+^o is the subvariety in $[\mathrm{Hom}(\mathbb{C}^6, \mathbb{C}^2)/\Gamma]$ where the image of Φ lies in one of the three roots of the cubic $t_1^3 + t_2^3$. This is three 6-dimensional linear subspaces meeting at the origin. So F is a singular orbifold, *i.e.* the quotient of a singular variety by a finite group.

Remark 3.3. Since Γ has 9 irreducible representations we have an equality of Euler characteristics

$$\chi(X_+) \;=\; \chi(F) \;=\; \chi(B\Gamma)+\chi(\mathbb{P}^5) \;=\; 9+6 \;=\; 15 \;=\; \chi(\mathrm{Gr}(2,6)) \;=\; \chi(X_-)$$

which is a necessary condition for derived equivalence.

Finally, we observe that F has a resolution by a weighted projective space:

$$\widetilde{F} \cong \mathbb{P}^6_{1:\ldots:1:6} \longrightarrow F \tag{3.4}$$

We construct this resolution by making an additional choice of a line $L \subset T$ that is a root of ψ, and which contains the image of Φ. Away from $\Phi = 0$, this is no extra data, but at the singular point it separates the three branches whilst also reducing the isotropy group from Γ to $\mathbb{Z}_6$. To see that the resulting resolution $\widetilde{F}$ really is the claimed weighted projective space, we set $L = \{t_2 = 0\}$ and choose ψ to be of the form:

$$\psi = t_2(t_1^2 + \gamma t_2^2)$$

The image of Φ lies in $\{t_2 = 0\}$, *i.e.* the bottom row of Φ is zero, and the residual symmetry is the 1-parameter subgroup λ_2. It acts on the top row of Φ with weight -1 and on the coefficient γ with weight -6.

3.5. **The critical locus.** Here we analyse the critical locus of the superpotential W on the stack $\mathcal{X}$. Recall that

$$W = (\psi \circ \mathrm{Sym}^3 \Phi)(f)$$

where $\psi \in \mathrm{Sym}^3 T^\vee$, $\Phi \in \mathrm{Hom}(\mathbb{C}^6, T)$, and f is a fixed cubic in 6 variables, the defining equation of our cubic fourfold Y.

First, observe that W is linear in ψ, so $\partial_\psi W = 0$ means $\mathrm{Sym}^3 \Phi(f) = 0$. This says that the pullback of the cubic f via Φ is the zero cubic in two variables. In particular, either

- $\Phi = 0$; or
- $\mathrm{rank}(\Phi) = 1$ and the image of $\mathrm{Sym}^3 \Phi$ is a point in Y; or
- $\mathrm{rank}(\Phi) = 2$ and the image of Φ is a point of F_Y.

Now we look at the other derivatives. We use the stratification Σ_i of $\mathrm{Sym}^3 T^\vee$ introduced above and deal with each stratum separately.

(Σ_0) If $\psi = 0$, then $W_\psi \equiv 0$, so the Φ derivatives vanish.

(Σ_1) If $\psi = t_1^3$, then $W_\psi = f(\Phi_{11}, \ldots, \Phi_{16})$. Since f is non-singular, the Φ derivatives vanish exactly at $\Phi_{1\bullet} = 0$, *i.e.* when $\mathrm{rank}\,\Phi \leq 1$ and the image of Φ is in the root of ψ.

(Σ_3) If $\psi = t_1^3 + t_2^3$, then $W_\psi = f(\Phi_{1\bullet}) + f(\Phi_{2\bullet})$, and the Φ derivatives vanish only at $\Phi = 0$.

(Σ_2) Set $\psi = t_1^2 t_2$. Write $\widehat{f}$ for the symmetric trilinear map corresponding to the cubic form f, so

$$W_\psi = \widehat{f}(\Phi_{1\bullet}, \Phi_{1\bullet}, \Phi_{2\bullet})$$

This is linear in the variables Φ_{2i}, and we have:

$$\partial_{\Phi_{2i}} W_\psi = \partial_i f(\Phi_{11}, \ldots, \Phi_{16})$$

Again these vanish exactly when $\Phi_{1\bullet} = 0$, *i.e.* when $\mathrm{rank}\,\Phi \leq 1$ and the image of Φ is in the double root of ψ. And since W_ψ is quadratic in the Φ_{1i} variables, the derivatives $\partial_{\Phi_{1i}} W_\psi$ also vanish at these points.

Now we compare this with our description of the semistable loci from Section 3.3. In X_- we have $\mathrm{rank}\,\Phi = 2$, so the only critical points are along

$$F_Y \subset \mathrm{Gr}(6,2) \subset X_-$$

This is a standard part of the Knörrer periodicity equivalence (2.3). More importantly, we see that in X_+, there is a single critical point

$$\{\Phi = 0\} \in X_+^o$$

since all the critical points with $\psi \in \Sigma_2$ are unstable. This is essential for the restriction functor (2.9) to be an equivalence.

4. The proof of Theorem B

We will now prove Theorem 2.5 by constructing tilting generators of $\mathrm{D^b}(X_-)$ and $\mathrm{D^b}(X_+)$ with the same endomorphism algebra. We prove the two sides separately as Propositions 4.5 and 4.7 below.

We will use the techniques of [ŠVdB] and the machinery of grade restriction subcategories [HL], *a.k.a.* windows, so most of our arguments take place on the linear stack $\mathcal{X}$. This approach helps to motivate our construction of the generator E_+ but it is not essential; the proofs could be written purely on X_+ and X_-.

4.1. **Grade restriction subcategories.** Recall that our GIT problem is the vector space

$$V = \mathrm{Hom}(\mathbb{C}^6, T) \oplus \mathrm{Sym}^3 T^\vee$$

with the action of the group:

$$G = GL(T) \cong GL_2$$

As before, we write $\mathcal{X} = [V/G]$ for the Artin quotient stack. We have two GIT stability conditions and hence two GIT quotients, which are open substacks $X_\pm = [V_\pm^{ss}/G] \subset \mathcal{X}$. The theory developed in [HL] provides us with two subcategories

$$\mathcal{G}_\pm \subset D^b(\mathcal{X})$$

that are equivalent to $D^b(X_\pm)$ via the natural restriction functors.

Recall (Section 3.3) that for the negative stability condition, unstable points are, up to the action of G, destabilized by one of the two 1-parameter subgroups λ_0, λ_1. This provides a *KN stratification* of the unstable locus. For each i we have a fixed subspace

$$Z_i = V^{\lambda_i}$$

and also an attracting subspace $V^{\lambda_i \geq 0}$ of points destabilized by λ_i, *i.e.* the sum of the eigenspaces for non-negative eigenvalues of λ_i. There is an obvious stratification of the unstable locus into locally closed pieces

$$GV^{\lambda_0 \geq 0} = V^{\lambda_0 \geq 0} = \{\Phi = 0\} \qquad \text{and} \qquad GV^{\lambda_1 \geq 0} \setminus V^{\lambda_0 \geq 0} = \{\mathrm{rank}\, \Phi = 1\}$$

but we also need to be precise about how this stratification interacts with the fixed and attracting subspaces.

Notation 4.1. Since we have fixed a basis of T, a point in V consists of a matrix Φ and a 4-dimensional vector ψ, which we will write as

$$\begin{pmatrix} \vec{\Phi}_1 \\ \vec{\Phi}_2 \end{pmatrix}, (\psi_1, \psi_2, \psi_3, \psi_4)$$

where the components $\vec{\Phi}_i$ are 6-vectors, the rows of Φ. In such a presentation, we will use $\square$ to denote a component that is nonzero, and $\star$ to denote a component that is either zero or nonzero.

With this notation in place, we introduce the locally closed subvarieties:

$$Z_0^- = Z_0 = \left\{ \begin{pmatrix} \vec{0} \\ \vec{0} \end{pmatrix}, (0,0,0,0) \right\} \qquad Y_0^- = V^{\lambda_0 \geq 0} = \left\{ \begin{pmatrix} \vec{0} \\ \vec{0} \end{pmatrix}, (\star,\star,\star,\star) \right\}$$

$$Z_1^- = Z_1 \setminus Y_0^- = \left\{ \begin{pmatrix} \vec{\square} \\ \vec{0} \end{pmatrix}, (\star,0,0,0) \right\} \qquad Y_1^- = V^{\lambda_1 \geq 0} \setminus Y_0^- = \left\{ \begin{pmatrix} \vec{\square} \\ \vec{0} \end{pmatrix}, (\star,\star,\star,\star) \right\}$$

The unstable locus on the negative side is $Y_0^- \sqcup GY_1^-$, and both pieces are smooth.

For the positive stability condition, we have a similar stratification of the unstable locus, except that (i) we have a third destabilizing 1-parameter subgroup λ_2 to consider, so the stratification has three pieces, and (ii) the destabilized points are now $V^{\lambda_i\leq 0}$ instead of $V^{\lambda_i\geq 0}$. Using the same notation as above, we have:

$$Z_0^+ = Z_0 = \left\{\left(\begin{smallmatrix}\vec{0}\\ \vec{0}\end{smallmatrix}\right), (0,0,0,0)\right\} \qquad Y_0^+ = V^{\lambda_0\leq 0} = \left\{\left(\begin{smallmatrix}\vec{\star}\\ \vec{\star}\end{smallmatrix}\right), (0,0,0,0)\right\}$$

$$Z_1^+ = Z_1 \setminus Y_0^+ = \left\{\left(\begin{smallmatrix}\vec{\star}\\ \vec{0}\end{smallmatrix}\right), (\square,0,0,0)\right\} \qquad Y_1^+ = V^{\lambda_1\leq 0}\setminus Y_0^+ = \left\{\left(\begin{smallmatrix}\vec{\star}\\ \vec{\star}\end{smallmatrix}\right), (\square,0,0,0)\right\}$$

$$Z_2^+ = Z_2 \setminus GY_1^+ = \left\{\left(\begin{smallmatrix}\vec{0}\\ \vec{0}\end{smallmatrix}\right), (0,\square,0,0)\right\} \qquad Y_2^+ = V^{\lambda_2\leq 0}\setminus GY_1^+ = \left\{\left(\begin{smallmatrix}\vec{0}\\ \vec{\star}\end{smallmatrix}\right), (\star,\square,0,0)\right\}$$

The unstable locus is $Y_0^+ \sqcup GY_1^+ \sqcup GY_2^+$, matching the description we gave in Section 3.3.

We also need the following integer invariants:

$$\eta_i = \sum \text{positive eigenvalues of } \lambda_i \text{ on } V \;-\; \sum \text{positive eigenvalues of } \lambda_i \text{ on } \mathfrak{g}$$

Note that because $\mathcal{X}$ satisfies the Calabi-Yau condition, replacing λ_i with $-\lambda_i$ in this definition gives the same answer. In terms of the statification, η_i computes the total λ_i-weight of the conormal bundle $N^\vee_{GY_i^\pm}V$. In our example, it is easy to compute:

$$\eta_0 = 12 \qquad \eta_1 = 6-1=5 \qquad \eta_2 = 15-3=12$$

Given an object $\mathcal{F}\in D^b(\mathcal{X})$, the homology sheaves

$$\mathcal{H}^\bullet(\mathcal{F}|_{Z_i^\pm})$$

of the restriction of $\mathcal{F}$ to $Z_i^\pm$ split into weight spaces for the λ_i action. We define the *grade restriction subcategories* $\mathcal{G}_\pm \subset D^b(\mathcal{X})$ to be the full subcategories of objects $\mathcal{F}$ that obey the following *grade restriction rules*:

$$\text{The } \lambda_i\text{-weights of } \mathcal{H}^\bullet(\mathcal{F}|_{Z_i^\pm}) \text{ lie in } \left[-\tfrac{1}{2}\eta_i,\, \tfrac{1}{2}\eta_i\right) \tag{4.2}$$

Here i runs over the pieces of the stratification, so $\mathcal{G}_-$ is defined by two grade restriction rules, and $\mathcal{G}_+$ by three.

Theorem 4.3. [HL] *The restriction functors $D^b(\mathcal{X}) \to D^b(X_\pm)$ induce equivalences:*

$$\mathcal{G}_\pm \xrightarrow{\sim} D^b(X_\pm)$$

In fact, there are many possible grade-restriction subcategories, since for each i one can shift the interval in (4.2) by an integer and the above theorem still holds. For our purposes, it is simplest to center the intervals on 0.

In general, it can be hard to determine whether an object lies in a grade restriction category, but there is a class of objects for which it is easy. Pick an irreducible representation of G, indexed by some dominant weight $\chi = (a,b)$. There is a corresponding vector bundle on the stack $\mathcal{X}$ which we denote by

$$\mathcal{T}_\chi := \mathrm{Sym}^{a-b}T \otimes (\det T)^b$$

To see if $\mathcal{T}_\chi$ satisfies (4.2) we just pair λ_i with both χ and its Weyl partner $\chi' = (b,a)$ and see if the answers lie in the right interval.

For $\mathcal{G}_-$ this gives four inequalities cutting out a set ∇_- of 15 weights, and for $\mathcal{G}_+$ it gives six inequalities cutting out a set ∇_+ of 14 weights. See Figure 1.

Thus, we have a set of 15 vector bundles:

$$\{\mathcal{T}_\chi\,,\ \chi\in\nabla_-\} = \left\{\mathrm{Sym}^c T\otimes(\det T)^b\,,\ c\leq 4,\quad -2\leq b\leq 2-c\right\} \tag{4.4}$$

on $\mathcal{X}$ which all lie in $\mathcal{G}_-$. We shall prove in the next section that the direct sum of the restriction of these bundles to X_- give a tilting generator of $D^b(X_-)$. Note that this is just the pullback to X_- of Kapranov's exceptional collection on $\mathrm{Gr}(6,2)$ [K].

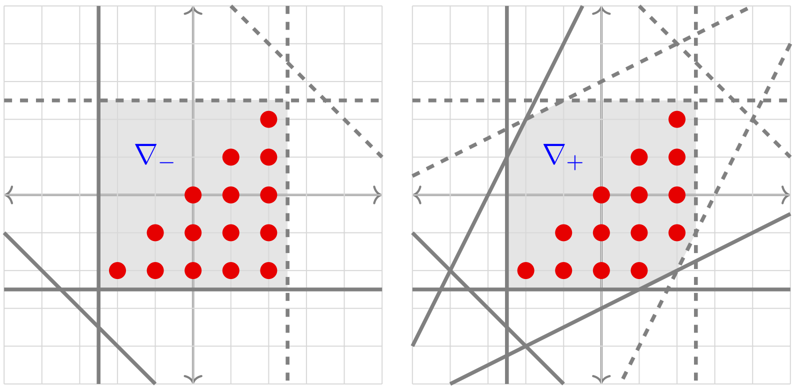

FIGURE 1. The red dots are the dominant weights corresponding to vector bundles that satisfy the grade restriction rules for $\mathcal{G}_-$ (left) and $\mathcal{G}_+$ (right).

Unfortunately, things are not so simple on X_+. The bundle

$$\mathcal{T}_{(2,-2)} = \operatorname{Sym}^4 T \otimes (\det T)^{-2}$$

fails the grade restriction rule for λ_2, so it does not lie in $\mathcal{G}_+$; and indeed one can show that the higher self-Exts of $\mathcal{T}_{(2,-2)}|_{X_+}$ do not vanish (Remark 4.14). Moreover, since the Euler characteristic of X_+ is 15 (Remark 3.3), the bundles from ∇_+ alone cannot generate $D^b(X_+)$. So we will have to work a bit harder to find a tilting generator on the positive side.

4.2. **Weyman's complexes.** Consider the locus $GV^{\lambda_1 \geq 0} = \{\operatorname{rank} \Psi \leq 1\}$, the closure of GY_1^-. This is a singular subvariety of V, and it has a Springer-type resolution:

$$r : G \times_P V^{\lambda_1 \geq 0} \longrightarrow GV^{\lambda_1 \geq 0}$$

Here, $P \subset G$ is the parabolic associated to λ_1, so $G/P = \mathbb{P}T \cong \mathbb{P}^1$. More explicitly, the resolution is the vector bundle:

$$G \times_P V^{\lambda_1 \geq 0} \;\cong\; \operatorname{Tot}\left\{ \operatorname{Hom}(\mathbb{C}^6, L) \oplus \operatorname{Sym}^3 T^\vee \longrightarrow \mathbb{P}T \right\}$$

where L is the tautological line bundle on $\mathbb{P}T$.

For any positive line bundle on this resolution, we have an associated sheaf $r_*(L^{-k})$ on $\mathcal{X}$, supported on $GV^{\lambda_1 \geq 0}$. Weyman [W] introduced a method for constructing free resolutions of such sheaves: we embed the resolution into $V \times \mathbb{P}T$, take the Koszul resolution of our sheaf, and push down. This produces a free resolution

$$C_{\lambda_1,k} \xrightarrow{\sim} r_*(L^{-k})$$

whose terms are sums of bundles $\mathcal{T}_\chi$, and the weights that appear are easily calculated by Borel-Weil-Bott. This method works for any 1-parameter subgroup $\lambda \subset G$, and indeed more generally for other representations of other reductive groups.[4]

These complexes were used to great effect by Špenko-Van den Bergh to construct non-commutative resolutions of quotient singularities [ŠVdB]. They are also relevant to GIT, since when we restrict $C_{\lambda_1,k}$ to X_- we get an exact sequence of vector bundles. In the right circumstances such sequences can be used to show the derived category of the quotient is generated by vector bundles, *e.g.* [HLS]. We can also view this as the process of mutating into the grade restriction category.

Now recall our set of 15 vector bundles (4.4) that lie in $\mathcal{G}_-$. We denote their sum by:

$$\mathcal{E}_- = \bigoplus_{\chi \in \nabla_-} \mathcal{T}_\chi$$

Proposition 4.5. *The vector bundle $E_- = \mathcal{E}_-|_{X_-}$ is tilting and its summands generate $D^b(X_-)$.*

[4]For larger G the resolution will be a vector bundle over some partial flag variety G/P.

Proof. From Theorem 4.3 we know:

$$\mathrm{Ext}^\bullet_{X_-}(E_-, E_-) = \mathrm{Ext}^\bullet_{\mathcal{X}}(\mathcal{E}_-, \mathcal{E}_-) = \mathrm{Ext}^0_{\mathcal{X}}(\mathcal{E}_-, \mathcal{E}_-)$$

On the Grassmannian itself, we know from [K] that every object can be resolved using the summands of E_-. It is a straight forward exercise to reprove this using the methods of [ŠVdB]; the complexes $C_{\lambda_1,k}$ produce enough exact sequences to reduce everything to ∇_-. The exact same proof also works for X_- since the $\mathrm{Sym}^3 T^\vee$ factor will have no effect on these constructions. □

As before, we denote the endomorphism algebra of E_- by

$$A = \mathrm{End}_{X_-}(E_-)$$

Then A is a non-commutative crepant resolution of X_0.

4.3. **The tilting sheaf on X_+.** Recall the subset of $\mathcal{X}$ that is destabilized by λ_2 on the negative side:

$$GV^{\lambda_2 \geq 0} = \{\mathrm{rank}\, \Phi < 2 \text{ and the image of } \Phi \text{ is a root of } \psi\}$$

(This did not appear in our KN stratification since such points are already destabilized by λ_1). It is a singular subvariety of V, and it has a resolution

$$q : G \times_P V^{\lambda_2 \geq 0} \longrightarrow GV^{\lambda_2 \geq 0}$$

Here, $P \subset G$ is the parabolic associated to λ_2, so $G/P = \mathbb{P}T \cong \mathbb{P}^1$, and explicitly:

$$G \times_P V^{\lambda_2 \geq 0} \;\cong\; \mathrm{Tot}\left\{ \mathrm{Hom}(\mathbb{C}^6, L) \oplus \mathrm{Sym}^2 T^\vee \otimes (T/L)^\vee \longrightarrow \mathbb{P}T \right\} \tag{4.6}$$

If we intersect with X_+ then $GV^{\lambda_2 \geq 0}$ is precisely the subset F discussed in Section 3.4, the fibre of X_+ over the origin in X_0. Moreover, this resolution is the same as our previous $\widetilde{F}$ (3.4).

Now consider the torsion sheaf $q_*(L^{-4})$ on $\mathcal{X}$. By adjunction, there is a canonical map:

$$\mathrm{Sym}^4 T^\vee \to q_*(L^{-4})$$

This is the final term of Weyman's free resolution. In particular, it is a surjection. We twist by a line bundle and take its kernel:

$$\mathcal{K} = \mathrm{Ker}\left(\mathcal{T}_{(2,-2)} \to q_*(L^{-4}) \otimes (\det T)^2 \right)$$

This, it turns out, is the correct modification of $\mathcal{T}_{(2,-2)}$. Notice that the restriction of $\mathcal{K}$ to X_- agrees with the bundle $\mathcal{T}_{(2,-2)}$, but the restriction of $\mathcal{K}$ to X_+ does not.

Proposition 4.7. *Let*

$$\mathcal{E}_+ = \mathcal{K} \oplus \bigoplus_{\chi \in \nabla_+} \mathcal{T}_\chi$$

and let $E_+ = \mathcal{E}_+|_{X_+}$. Then E_+ is a tilting sheaf, its summands generate $D^b(X_+)$, and it has endomorphism algebra:

$$\mathrm{End}_{X_+}(E_+) = A$$

Proof. We will prove below the following claims:

(1) $\mathcal{K}$ lies in the grade restricted subcategory $\mathcal{G}_+$ (Lemma 4.9).

(2) We have
$$\mathrm{End}(\mathcal{E}_+) = \mathrm{End}(\mathcal{E}_-)$$
and $\mathrm{Ext}^k(\mathcal{E}_+, \mathcal{E}_+) = 0$ for $k > 0$ (Lemma 4.8).

(3) The sheaf E_+ spans $D^b(X_+)$ (Proposition 4.11).

From (1) it follows that $\mathrm{Ext}^\bullet_{X_+}(E_+, E_+) = \mathrm{Ext}^\bullet_{\mathcal{X}}(\mathcal{E}_+, \mathcal{E}_+)$, and from (2) this is the algebra A. So the summands of E_+ generate a subcategory of $D^b(X_+)$ which is equivalent to $D^b(A)$ and hence admissible (since A is smooth). By (3) this subcategory is the whole of $D^b(X_+)$. □

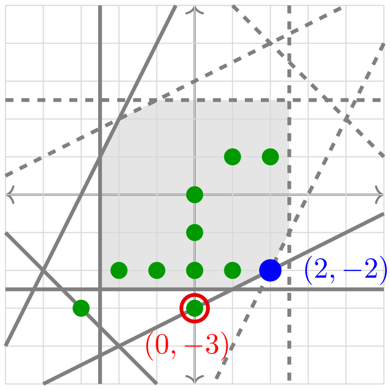

FIGURE 2. The bundles appearing in the free resolution of $\mathcal{K}$ (in green) together with $\mathcal{T}_{(2,-2)}$ (in blue). The final term of the resolution is circled in red. The grade restriction rules for $\mathcal{G}_+$ are also shown.

It follows from above that the summands of $\mathcal{E}_+$ generate $\mathcal{G}_+$, and that the restriction functor from $\mathcal{G}_+$ to $D^b(X_-)$ is an equivalence. In particular, our derived equivalence between X_+ and X_- can also be expressed as the composition:

$$D^b(X_+) \xleftarrow{\sim} \mathcal{G}_+ \xrightarrow{\sim} D^b(X_-)$$

Thus $\mathcal{G}_+$ provides the correct 'window' for this derived equivalence.

Note that we cannot replace $\mathcal{G}_+$ with $\mathcal{G}_-$ here because the restriction functor from $\mathcal{G}_-$ to $D^b(X_+)$ is not an equivalence (Remark 4.14).

4.4. **More on $\mathcal{K}$.** Using Weyman's method, the torsion sheaf $q_*(L^{-4}) \otimes (\det T)^2$ has a free resolution by bundles of the form $\mathcal{T}_\chi$. The weights that appear are easily calculated and are shown in Figure 2. The first term of the resolution is $\mathcal{T}_{(2,-2)}$, so by deleting that term, we obtain a free resolution of the sheaf $\mathcal{K}$.

Note that the final term of the resolution is $\mathcal{T}_{(0,-3)}$ (with multiplicity one), and this bundle does not appear in the other terms.

Lemma 4.8.

$$\mathrm{Hom}(\mathcal{E}_+, \mathcal{E}_+) = \mathrm{Hom}(\mathcal{E}_-, \mathcal{E}_-)$$

and $\mathrm{Ext}^k(\mathcal{E}_+, \mathcal{E}_+) = 0$ *for* $k > 0$.

Proof. Write $\mathcal{S} = q_*(L^{-4}) \otimes (\det T)^2$, so by definition we have a short exact sequence:

$$\mathcal{E}_+ \to \mathcal{E}_- \to \mathcal{S}$$

Our claim is that:

$$\text{(i)}\ \ \mathrm{Ext}^\bullet(\mathcal{E}_+, \mathcal{S}) = 0 \qquad \text{and} \qquad \text{(ii)}\ \ \mathrm{Ext}^\bullet(\mathcal{S}, \mathcal{E}_-) = 0$$

For (i), we claim more precisely that $\mathcal{S}$ is right orthogonal to $\mathcal{T}_\chi$ for every χ in ∇_+ and every χ appearing in our free resolution of $\mathcal{K}$. This holds by the first statement of [ŠVdB, Lem. 11.2.1] since every such χ satisfies the inequality:

$$\langle \lambda_2, \chi \rangle \ > \ \langle \lambda_2, (2,-2) \rangle$$

Part (ii) is the claim that $\mathcal{S}$ is left orthogonal to $\mathcal{T}_\chi$ for every $\chi \in \nabla_-$. We use the fact [W] that $\mathcal{S}^\vee$ is a shift of:

$$q_*((T/L)^3)$$

Then left orthogonality to ∇_+ follows as above, since ∇_+ is invariant under

$$\mathcal{T}_{(a,b)} \mapsto \mathcal{T}^\vee_{(a,b)} = \mathcal{T}_{(-b,-a)}$$

and every $\chi \in \nabla_+$ satisfies $\langle \lambda_2, \chi \rangle > \langle \lambda_2, (3,0) \rangle$. Finally, to get that $\mathrm{Ext}^\bullet(\mathcal{S}, \mathcal{T}_{(2,-2)}) = 0$ we need to use the rest of [ŠVdB, Lem. 11.2.1], and observe that every weight in $(a,b) \in (V^{\lambda_2 \geq 0})^\vee$ satisfies $a > 0$. $\square$

Lemma 4.9. *$\mathcal{K}$ lies in the grade restriction category $\mathcal{G}_+$.*

Proof. It is clear from Figure 2 that $\mathcal{K}$ satisfies the required grade restriction rules for λ_0 and λ_2 on both sides because all the terms in the free resolution lie in the correct region. But this is not true for λ_1; in fact, $\mathcal{K}$ cannot satisfy the λ_1 grade restriction rule on the negative side. If it did, then the torsion sheaf $\mathcal{S}$ would lie in $\mathcal{G}_-$, which would be a contradiction since $\mathcal{S}$ restricts to zero in X_-.

However, for $\mathcal{K}$ to lie in $\mathcal{G}_+$, we only require that $\mathcal{S}$ (and hence also $\mathcal{K}$) satisfies the λ_1 grade restriction rule along the locally closed subset:

$$Z_1^+ = V^{\lambda_1} \setminus \{\psi \in \Sigma_1\}$$

We verify this with an explicit computation.

We are interested in the torsion sheaves between $\mathcal{O}_{Z_1}$ and $\mathcal{S} = q_*(L^{-4}) \otimes (\det T)^2$, and more specifically in the λ_1 weights that occur in them. Recall, in the notation of Section 4.1, that

$$Z_1 = \left\{ \left(\begin{smallmatrix} \vec{\star} \\ \vec{0} \end{smallmatrix}\right), (\star, 0, 0, 0) \right\} \qquad \text{and} \qquad Z_1^+ = \left\{ \left(\begin{smallmatrix} \vec{\star} \\ \vec{0} \end{smallmatrix}\right), (\square, 0, 0, 0) \right\}$$

Using the evident projective coordinates on the resolution (4.6), we can write the map q as:[5]

$$q : (x_1, ..., x_6 \,|\, \alpha, \beta, \gamma \,|\, u\!:\!v) \mapsto \begin{pmatrix} ux_1 & \dots & ux_6 \\ vx_1 & \dots & vx_6 \end{pmatrix}, \ (v\alpha, v\beta - u\alpha, v\gamma - u\beta, -u\gamma)$$

We care about the λ_1 grading, which on the source coordinates we can take to be

$$1, ..., 1 \,|\, 0, -1, -2 \,|\, -1, 0$$

(we have some freedom here because these are projective coordinates). Since we only care about Z_1^+, we can set $v = 1$ and $\alpha \neq 0$, so q becomes a map from $\mathbb{C}^9 \times \mathbb{C}^*$ to $\mathbb{C}^{15} \times \mathbb{C}^*$. The line bundle L is trivial on this locus, so $\mathcal{S}$ is the pushdown of the free graded module $\mathcal{O}(2)$.

To compute the torsion between $q_*\mathcal{O}$ and $\mathcal{O}_{Z_1}$, we take the Koszul resolution of the latter and pull it back via q. This produces the Koszul complex for the nine elements

$$x_1, ..., x_6, \beta - u\alpha, \ \gamma - u\beta, \ -u\gamma$$

in the graded ring $\mathbb{C}[x_1, ..., x_6, \alpha^{\pm 1}, \beta, \gamma, v]$. This is a regular sequence; the vanishing locus is a (singular) curve so it has the expected dimension. Hence

$$\mathrm{Tor}^\bullet(q_*\mathcal{O}, \mathcal{O}_{Z_1^+}) = \mathrm{Tor}^0(q_*\mathcal{O}, \mathcal{O}_{Z_1^+}) = \mathbb{C}[\gamma^{\pm 1}, u]/u^3 \ = \ \mathcal{O}_{Z_1}[u]/(x_1, ..., x_6, u^3)$$

The λ_1 weight of the variable u is -1 so the weights that occur here are exactly $0, -1, -2$. It follows that $\mathcal{S}$ satisfies the λ_1 grade restriction rule. $\square$

Remark 4.10. As a check on this calculation, we can look instead at the locus $Z_1^- = Z_1 \setminus \{\Psi = 0\}$. This leads us to the Koszul complex for

$$vx_1, ..., vx_6, v\beta - \alpha, \ v\gamma - \beta, \ -\gamma$$

over the ring $\mathbb{C}[x_1, ..., x_6, \alpha, \beta, \gamma, v]$. This is not a regular sequence, and it has the same homology as the Koszul complex of $(vx_1, ..., vx_6)$. The λ_1 weights here are $0, ..., -5$ violating the grade restriction rule as expected.

4.5. **Generation on X_+.** We now prove our final required fact, that the sheaf E_+ spans the category $D^b(X_+)$.

Proposition 4.11. *If $\mathcal{F} \in D^b(X_+)$ satisfies*

$$\mathrm{Ext}^\bullet(E_+, \mathcal{F}) = 0$$

then $\mathcal{F} \simeq 0$.

[5]The components of the 4-vector here are the coefficients of the cubic $(vt_1 - ut_2)(\alpha t_1^2 + \beta t_1 t_2 + \gamma t_2^2)$ that vanishes on the line $u\!:\!v$.

Proof. Recall that X_+ contains an open set

$$X_+^o \cong [\mathrm{Hom}(\mathbb{C}^6, \mathbb{C}^2) \,/\, \Gamma]$$

and the complement is the divisor D (3.1), which is a bundle over $\mathbb{P}^5$. We denote the two inclusions by:

$$X_+^o \xhookrightarrow{\ f\ } X_+ \xhookleftarrow{\ g\ } D$$

The local cohomology of $\mathcal{F}$ along D fits into a triangle:

$$R\Gamma_D(\mathcal{F}) \to \mathcal{F} \to f_* f^* \mathcal{F}$$

Now consider the subcategory $\langle E_+ \rangle \subset D^b(X_+)$ generated by the summands of E_+. By assumption, $\mathcal{F}$ is orthogonal to this subcategory. By Lemma 4.12 below, $\langle E_+ \rangle$ contains all objects that are pushed-forward from D, and hence it contains the whole subcategory $D^b_D(X_+)$ of objects supported on D. It follows that $R\Gamma_D(\mathcal{F}) = 0$.

The irreducible representations of G indexed by ∇_+, when restricted to Γ, provide us with all nine irreducible representations of Γ (see Section 3.2 and fact (iii) in particular). So f^*E_+ spans $D^b(X_+^o)$ and hence we also have $f_* f^* \mathcal{F} = 0$. $\square$

Lemma 4.12. *The image of the functor $g_* : D^b(D) \to D^b(X_+)$ lies in the subcategory $\langle E_+ \rangle$ generated by the summands of E_+.*

Proof. The divisor D is cut out by a section of the line bundle $(\det T)^6 = \mathcal{T}_{(6,6)}$ (Remark 3.2), so we have a short exact sequence:

$$\mathcal{T}_{(-6,-6)} \to \mathcal{O}_{X_+} \to g_* \mathcal{O}_D$$

Moreover, $D^b(D)$ is generated by the six line bundles $\mathcal{O}_D, ..., \mathcal{O}_D(5)$. So we are reduced to showing that the line bundles

$$\mathcal{T}_{(k,k)} = \det(T)^k, \quad \text{for } k \in [-6, 5]$$

lie in $\langle E_+ \rangle$.

Recall the locus $GV^{\lambda_1 \leq 0}$ destabilised by λ_1 on the positive side. This is the set where ψ has a triple root, *i.e.* $\mathrm{Hom}(\mathbb{C}^6, T) \times \Sigma_1$. Ignoring the linear factor, its resolution is the line bundle $(T/L)^{-3}_{\mathbb{P}T}$. For every positive line bundle on the resolution, we get an exact sequence of vector bundles on X_+, of which the first three are:

$$\mathcal{T}_{(5,4)} \longrightarrow \mathcal{T}_{(4,2)} \longrightarrow \mathcal{O}$$

$$\mathcal{T}_{(5,5)} \longrightarrow \mathcal{T}_{(3,1)} \longrightarrow \mathcal{T}_{(1,0)}$$

$$\mathcal{T}_{(5,3)} \to \mathcal{T}_{(4,1)} \to \mathcal{T}_{(2,0)} \tag{4.13}$$

(the first two are dual up to a twist). Starting from the set ∇_+, we can use twists of these three sequences to recursively generate the bundles $\mathcal{T}_\chi$ for the weights shown in Figure 3 (L), which include $\mathcal{T}_{(k,k)}$ for $k \in [-5, 5]$.

Now compare this with our free resolution of $\mathcal{K}$ (Figure 2). Since the final term is $\mathcal{T}_{(0,-3)}$ we have a cone $[\mathcal{K} \to \mathcal{T}_{(0,-3)}]$ and this consists of bundles that we have already generated. So, by including $\mathcal{K}$ as well, we can generate $\mathcal{T}_{(0,-3)}$. See Figure 3 (R). Then two more applications of the sequences (4.13) give us $\mathcal{T}_{(-2,-4)}$ and finally $\mathcal{T}_{(-6,-6)}$. $\square$

Remark 4.14. Consider the locus $GV^{\lambda_2 \leq 0}$ destabilized by λ_2 on the positive side. By pushing down the right line bundle from the resolution of this locus, we can produce an exact sequence of bundles on X_+ which both starts and ends with $\mathcal{T}_{(2,-2)}$. It represents a non-zero class in

$$\mathrm{Ext}^7_{X_+}\big(\mathcal{T}_{(2,-2)}, \mathcal{T}_{(2,-2)}\big).$$

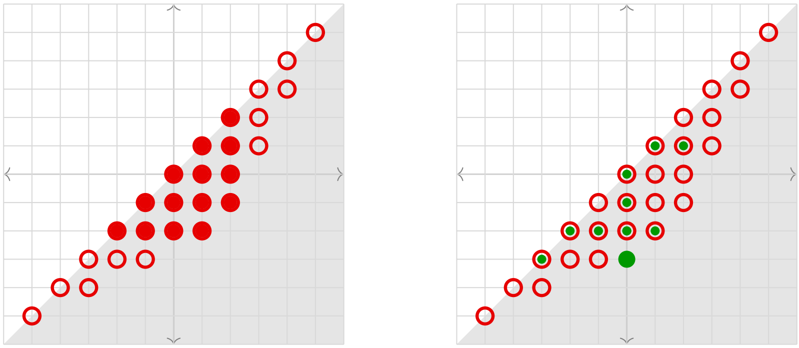

FIGURE 3. (L) Bundles that can be generated from ∇_+ using the sequences (4.13) alone. (R) Generating $\mathcal{T}_{(0,-3)}$ from the previous set and $\mathcal{K}$.

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